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**LETTER TO THE EDITOR**

# How to detect integrability in cellular automata

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**Abstract**

Ultra-discrete equations are generalized cellular automata in the sense that the dependent (and independent) variables take only integer values. We present a new method for identifying integrable ultra-discrete equations which is the equivalent of the singularity confinement property for difference equations and the Painlevé property for differential equations. Using this criterion, we find integrable ultra-discrete equations which include the ultra-discrete Painlevé equations.

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Integrable dynamical systems play universal roles as models of natural phenomena. They are valuable models because they possess no chaos and their solutions allow prediction. In continuous time, these systems include the famous soliton equations appearing in many physical contexts [1], such as fluids, plasma physics and optics. Reductions of such equations lead to the Painlevé equations, which appear in crucial roles in several exactly solvable statistical-mechanics models [2] and in random matrix theory [3]. Their discrete versions, which also appear in statistical mechanics, orthogonal-polynomial theory, several numerical algorithms [4] and random matrix theory, have been a focus of intense interest in the past 15 years [5].

Cellular automata (CA) have been widely adopted in the sciences as simple but powerful models of the real world because the complex patterns produced by their long-time behaviours can mimic observations with tremendous accuracy [6]. However, the lack of mathematical tools makes prediction difficult in CA models. That there are integrable, predictable CAs, possessing solitons, was confirmed by the beautiful work of Tokihiro *et al* [7]. They showed that integrable CA with soliton solutions may be obtained from well-known integrable equations such as the Korteweg–de Vries (KdV) equation. The path they took was through ultra-discrete equations. The aim of this letter is to present a new method of identifying integrable ultra-discrete equations, and hence integrable CA.

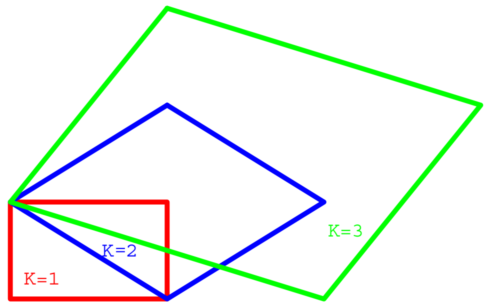


**Figure 1.** Curves joining iterates of equation (3) in the $(X_n, X_{n+1})$-plane, with $X_0 = 0$, $X_1 = 1$ and $K = 1$, $K = 2$, $K = 3$.

Ultra-discrete equations are obtained by a limiting process from discrete equations in a way that allows both the dependent and independent variables to take only discrete values. The first integrable ultra-discrete equations called *soliton cellular automata* were obtained by Takahashi *et al* [16, 17]. These are governed by filter parity rules and also related to box and ball systems. A method to ultra-discretize integrable systems was developed in [7, 18] followed by the study of different ultra-discrete versions of known integrable equations including the Painlevé equations [19–23]. One open problem was the lack of an algorithmic method for finding new integrable ultra-discrete equations. Our paper addresses this problem.

The crucial step in the discovery of the Painlevé equations [8] was the test for the Painlevé property, i.e., that all movable singularities of all solutions are poles. This *Painlevé test* has been used [9] repeatedly to obtain necessary conditions for integrability. In the discrete setting, there exist several tools to test an equation for integrability [10–15]. In particular, *singularity confinement* [10, 11], like the Painlevé test for continuous systems, provides a way to identify integrability through the study of the singularity of the solutions of a discrete system. In this paper, we extend the test to ultra-discrete equations.

For each variable (or parameter) $v$ in a given equation, the ultra-discretization method requires that we introduce a new variable $V$ defined by $v = \mathrm{e}^{\frac{V}{\epsilon}}$. Then we take the limit $\epsilon \to 0^+$ of the equation using the identity

$$\lim_{\epsilon\to 0^+} \epsilon \log\left(\mathrm{e}^{\frac{A}{\epsilon}} + \mathrm{e}^{\frac{B}{\epsilon}}\right) = \max(A, B). \tag{1}$$

Consider the discrete equation

$$x_{n+1}x_{n-1} = k + \frac{1}{x_n}, \tag{2}$$

where $k$ is constant. Under the ultra-discrete limit, this becomes

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0). \tag{3}$$

For integer $K$, and initial values $X_0$, $X_1$, all iterates are integer. Equation (2) is part of the *QRT* family [24] of integrable mappings and admits the following conserved quantity:

$$I = x_n + x_{n-1} + \frac{k}{x_n} + \frac{k}{x_{n-1}} + \frac{1}{x_n x_{n-1}}. \tag{4}$$

The corresponding conserved quantity for equation (3) is

$$I = \max(X_n, X_{n-1}, K - X_{n-1}, K - X_n, -X_n - X_{n-1}). \tag{5}$$

The phase-plot in figure 1 shows the qualitative nature of the invariant curves.

Discrete Painlevé equations have been found by using the singularity confinement test on non-autonomous versions of the QRT family. In the following, we show that our test is capable of producing ultra-discrete Painlevé equations.

The only singularity in equation (2) is $x = 0$ and it can be shown that iterates that come close to this singularity are confined in the sense that they are non-singular after a finite set of steps and are analytic in the initial data. Moreover, other integrability criteria, namely, conditions on the Nevanlinna order of the mapping and degree of growth of the mapping, are satisfied. The very valuable insight developed in [10, 11] was to use this criterion on the de-autonomized equation, where the constant coefficient $k$ is replaced by a function of $n$:

$$x_{n+1}x_{n-1} = \phi(n) + \frac{1}{x_n}.$$

By demanding singularity confinement, they found that the admissible equations in this class are given by [25] $\phi(n) = kq^n$, where $k$ and $q$ are constants.

In order to analyse (3), we consider the value $X_n = -K$ at which the right-hand side is not differentiable and study the iterates. To do so, we perform a local analysis by perturbing the point $X_n = -K$ by the nonzero real small $\epsilon$ and consider the case when $X_{n-1} > 2|K|$. Then the iterates are

| | $\epsilon > 0$ | $\epsilon < 0$ |
|---|---|---|
| $X_n$ | $-K + \epsilon$ | $-K + \epsilon$ |
| $X_{n+1}$ | $K - X_{n-1}$ | $K - X_{n-1} - \epsilon$ |
| $X_{n+2}$ | $X_{n-1} - \epsilon$ | $X_{n-1}$ |
| $X_{n+3}$ | $X_{n-1}$ | $X_{n-1} + \epsilon$ |
| $X_{n+4}$ | $K - X_{n-1} + \epsilon$ | $K - X_{n-1}$ |
| $X_{n+5}$ | $-K - \epsilon$ | $-K - \epsilon$ |
| $X_{n+6}$ | $X_{n-1}$ | $X_{n-1}$ |

From the table above one sees that $X_n, X_{n+1}, X_{n+2}, X_{n+3}, X_{n+4}$ are not differentiable at $X_n = -K$ since the coefficients of $\epsilon$ in the two different columns do not match. However, the differentiability is recovered at the next two steps for $X_{n+5}$ and $X_{n+6}$ and we argue that this behaviour characterizes integrability. A convincing argument in favour of our claim comes from considering the non-autonomous version of (3)

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0), \tag{6}$$

where now $\phi_n$ is an arbitrary function of $n$. The iterates read

| | $\epsilon > 0$ | $\epsilon < 0$ |
|---|---|---|
| $X_n$ | $-\phi_n + \epsilon$ | $-\phi_n + \epsilon$ |
| $X_{n+1}$ | $\phi_n - X_{n-1}$ | $\phi_n - X_{n-1} - \epsilon$ |
| $X_{n+2}$ | $X_{n-1} - \epsilon$ | $X_{n-1}$ |
| $X_{n+3}$ | $X_{n-1} - \phi_n + \phi_{n+2}$ | $X_{n-1} - \phi_n + \phi_{n+2} + \epsilon$ |
| $X_{n+4}$ | $\phi_{n+3} - X_{n-1} + \epsilon$ | $\phi_{n+3} - X_{n-1}$ |
| $X_{n+5}$ | $-\phi_{n+3} - \phi_{n+2} + \phi_n - \epsilon$ | $-\phi_{n+3} - \phi_{n+2} + \phi_n - \epsilon$ |

(where it has been assumed that $X_{n-1} > \max(\phi_n + \phi_{n+1}, -\phi_{n+2}, -\phi_{n+3} - \phi_{n+2} + \phi_n, \phi_{n+3} + \phi_{n+4})$. As before, $X_{n+5}$ is differentiable but for $X_{n+6}$ to be differentiable, $\phi_n$ must satisfy the equation

$$\phi_{n+5} - \phi_{n+3} - \phi_{n+2} + \phi_n = 0, \tag{7}$$

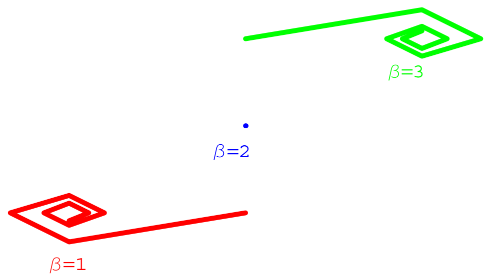


**Figure 2.** Curves joining scaled iterates of equation (6), with $\phi(n) = \beta n$, in the $(X_n/n, X_{n+1}/(n+1))$-plane, with $X_0 = 0$, $X_1 = 1$ and $\beta = 1$, $\beta = 2$, $\beta = 3$.

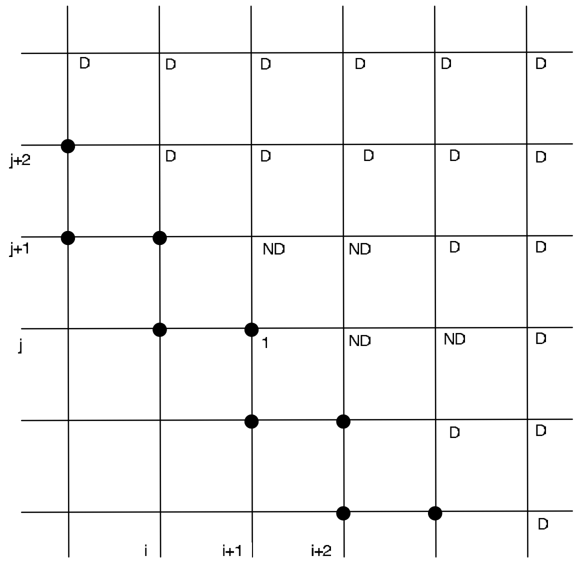


**Figure 3.** Figure showing the evolution of a singularity of (9) in the case $\sigma = 1$. The dots denote the locations at which arbitrary boundary conditions have been imposed (except for $u_j^{i+1} = 1+\epsilon$ which induces non-differential iterates as $\epsilon \to 0$). The D and ND stand, respectively, for differentiable and non-differentiable. The figure shows that the ND points are localized in the lattice and do not propagate.

whose general solution is

$$\phi_n = \alpha + \beta n + \gamma(-1)^n + \delta\cos\left(\frac{2\pi n}{3}\right) + \omega\sin\left(\frac{2\pi n}{3}\right), \tag{8}$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are arbitrary constants. No other conditions arise on $\phi_n$ from the remaining initial conditions. The phase-plot in figure 2 shows the qualitative nature of the orbits, for the case $\phi(n) = \beta n$. For $\gamma = \delta = \omega = 0$, one obtains a well-known ultra-discrete version of the first Painlevé equation (u-P$_{I-2}$ in [20]). When $\gamma$ is also nonzero, it corresponds to an ultra-discrete version of a degenerate form of the scaled discrete asymmetric version of the third Painlevé equation found in [11] (see equation (8) of this paper) which reads

$$\overline{y}\underline{y} = \delta_0^{\pm} q^n + \frac{1}{y},$$

where $\delta_0^{\pm}$ is a constant, the $\pm$ sign being dependent on the parity of $n$. Note that it is a straightforward exercise to show that $\delta$ and $\omega$ can always be brought to zero by the gauge transformation $X_n = \tilde{X}_n - \psi_n$ with $\psi_n = \left(\delta \cos\left(\frac{2\pi n}{3}\right) + \omega \sin\left(\frac{2\pi n}{3}\right)\right)$.

Using the same idea on other integrable autonomous discrete equations that are part of the QRT mappings, one can obtain all the known ultra-discretizations of Painlevé equations together with other new ones. Our extensive study of these will be published elsewhere [26].

Furthermore, our criterion can be applied to lattice equations evolving with time. For example, consider the equation

$$u_{j+1}^{i+1} = u_j^i + \max\left(u_j^{i+1} - 1, 0\right) - \sigma \max\left(u_{j+1}^i - 1, 0\right), \tag{9}$$

where $i$ is the discretization of time and $j$ is the discretization of space and $\sigma$ is a constant. The case $\sigma = 1$ corresponds to an ultra-discretization of the KdV equation and it was shown to admit $N$ soliton solutions [7]. The equation admits a singularity if $u_j^{i+1} = 1$. This situation, in the case $\sigma = 1$, is illustrated in figure 3 where it is shown that the singularity does not propagate in the two-dimensional plane for the class of initial conditions satisfying $u_j^i - \max\left(u_{j+1}^i - 1, 0\right) < 1$ and $u_{j-1}^{i+1} - \max\left(u_{j-1}^{i+2} - 1, 0\right) > \max\left(1, u_{j-1}^{i+2} - \max\left(u_{j-1}^{i+3} - 1, 0\right)\right)$. Note that other values of $\sigma$ do not give rise to the pattern illustrated in figure 3. For example, if $\sigma \neq 1$, $u_{j+2}^{i+2}$ and $u_{j+1}^{i+3}$ are not differentiable and the singularity appears to propagate through the plane. Our criterion is thus able to single out integrable equations in 1+1 dimensions. Note finally that our analysis carries over to CA (in which the dependent variable is restricted to take the values 0 or 1) such as that associated with (9) (equation (8) of [7]).

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